CMPUT 654 Fa 23: Theoretical Foundations of Machine Learning **Fall 2023**

Lecture 6: Small Risk Bound, Empirical Processes, Lower Bracketing (Sept 11)

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This lecture's notes illustrate some uses of various LaTeX macros. Take a look at this and imitate.

6.1 Recap

Recall the setting of ERM introduced in the previous lectures. We have a dataset (or datalist) $D_n = \{(X_i, f_*(X_i))\}_{i=1}^n$ where $X_i \sim P \in \mathcal{M}_1(\mathcal{X})$ are independent and $f_* \in C_d \subset \underline{2}^{2^d}$. Let $|C_d| = N < \infty$. For a fixed function $f \in \underline{2}^{2^d}$, let $L_n(f) = \sum_{i=1}^n \mathbb{I}(f(X_i) \neq f_*(Y_i))$ and $L(f) = \mathbb{E}[\mathbb{I}(f(X) \neq f_*(X))]$ for $X \sim P$. The empirical risk minimizer is $f_n = \arg\min_{f \in C_d} L_n(f)$. We used the multiplicative Chernoff bound to obtain the following proposition:

Proposition 6.1. *For $\delta \in (0,1)$, $f \in \underline{2}^{2^d}$ and $n, N \in \mathbb{N}$, let $\beta_\delta^n(f,N) = \sqrt{\frac{2L(f)\log(\frac{N}{\delta})}{n}}$. For all $f_0 \in C_d$ and $\delta \in (0,1)$, let $U(\delta, f_0, C_d)$ be the event that:*

$$U(\delta, f_0, C_d) := \left\{\forall f \in C_d : L(f) \leq L_n(f) + \beta_\delta^n(f, N+1)\right\} \bigcap \left\{L_n(f_0) \leq L(f_0) + \beta_\delta^n(f_0, N+1) + \frac{\log(\frac{N+1}{\delta})}{3n}\right\}.$$

It follows that $\mathbb{P}(U(\delta, f_0, C_d)) \geq 1 - \delta$.

For all $f_0 \in C_d$, on the event $U(\delta, f_0, C_d)$, we have that:

$$\begin{aligned} L(f_n) &\leq L_n(f_n) + \beta_\delta^n(f_n, N+1) \\ &\leq L_n(f_0) + \beta_\delta^n(f_n, N+1) && (f_n \text{ is the sol. to ERM}) \\ &\leq L(f_0) + \beta_\delta^n(f_0, N+1) + \beta_\delta^n(f_n, N+1) + \frac{\log(\frac{N+1}{\delta})}{3n}, \end{aligned}$$

which gives us the following theorem:

Theorem 6.2. *For all $f_0 \in C_d$, w.p. $1-\delta$,*

$$L(f_n) \leq L(f_0) + \beta_\delta^n(f_0, N+1) + \beta_\delta^n(f_n, N+1) + \frac{\log(\frac{N+1}{\delta})}{3n}.$$

Since the above theorem holds for all $f_0 \in C_d$, we can take the infimum:

Corollary 6.3. *w.p. $1-\delta$,*

$$L(f_n) \leq \beta_\delta^n(f_n, N+1) + \frac{\log(\frac{N+1}{\delta})}{3n} + \inf_{f \in C_d(\delta)} (L(f) + \beta_\delta^n(f, N+1))$$

Remark 6.4. In our current setting, $\inf_{f \in C_d(\delta)}(L(f) + \beta_\delta^n(f, N+1)) = 0$ because $L(f_*) + \beta_\delta^n(f_*, N+1) = 0$. Corollary 6.3 cannot buy us anything more than the bound we got in the last class because there is still a factor of $\sqrt{1/n}$ in $\beta_\delta^n(f_n, N+1)$. However, in more general settings where $L(f_*) \neq 0$, i.e., noises are injected to $f_*(X_i)$, we may get some benefit from Corollary 6.3.

6.2 Empirical Process

Now consider an arbitrary function class $\mathcal{F} \subset \mathcal{Y}^{\mathcal{X}}$ which is potentially infinite and an arbitrary (measurable) loss function $\ell : \mathcal{Y}\times\mathcal{Y} \to \mathbb{R}$ (instead of the 0-1 loss we considered in the previous section). Let $f_n = \arg\max_{f\in\mathcal{F}} L_n(f)$ be the empirical risk minimizer on $\mathcal{F}$. If we were to apply the technique in Proposition 6.1, the term $L_n(f) - L(f)$ for some $f \in \mathcal{F}$, would be the quantity that we would like to bound. To do that, one of the options is to bound:

$$\sup_{f\in\mathcal{F}} |L_n(f) - L(f)| = \sup_{f\in\mathcal{F}} \frac{1}{n} \left| \sum_{i=1}^{n} \ell(f(X_i), Y_i) - \int \ell(f(x), y)P(dx, dy) \right| \tag{6.1}$$

To reduce clutter, we define $D_i : \mathcal{F} \to \mathbb{R}$ for $i \in \mathbb{N}$ such that

$$D_i(f) = \ell(f(X_i), Y_i) - \int \ell(f(x), y)P(dx, dy),$$

and $\bar{D}_n : \mathcal{F} \to \mathbb{R}$ such that

$$\bar{D}_n(f) = \frac{1}{n}\sum_{i=1}^{n} D_i(f), \quad \forall f \in \mathcal{F}.$$

Note that $D_1(f), D_2(f), ...$ are i.i.d. random variables. Then Eq. (6.1) can be written as:

$$\sup_{f\in\mathcal{F}} \bar{D}_n(f).$$

We call $\{\bar{D}_n(f)\}_{n=1}^{\infty}$ an empirical process. Empirical process theory is a subarea of probability theory that studies the question of convergence of the process to 0 in different ways, e.g., convergence in probability or almost sure convergence. If $\bar{D}_n(f) \to 0$ in probability, it is called the *Weak Law of Large Number* and when $\sup_{f\in\mathcal{F}} \bar{D}_n(f) \to 0$ happens, we say that *uniform convergence* happens.

6.3 Lower Bracketing Number

Now we further reduce the clutter by introducing new notations. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$

$$G = \{(x, y) \to \ell(f(x), y) : f \in \mathcal{F}\} \subseteq \mathbb{R}^{\mathcal{X}\times\mathcal{Y}} = \mathbb{R}^{\mathcal{Z}}.$$

Let $Z_1, Z_2, ...Z_n \sim P \in \mathcal{M}_1(\mathcal{Z})$ and let $P_n(dz) = \frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i}(dz)$ be the *empirical distribution* where $\delta_{Z_i}(\{z\}) = 1$ if $z = Z_i$ and 0 otherwise. Note that δ_{Z_i} is a random measure. For $P \in \mathcal{M}_1(\mathcal{Z})$, let $Pg := \int gdP$ for $g \in \mathcal{G}$. Then Eq. (6.1) can be written as:

$$\sup_{g\in\mathcal{G}} |P_n g - Pg|$$

Definition 6.5. Let $\mathcal{G} \subseteq \mathbb{R}^{\mathcal{Z}}$ and fix $P \in \mathcal{M}_1(\mathcal{Z})$. For a fixed ε, $g_1, ...g_m \in \mathbb{R}^{\mathcal{Z}}$ is called a lower bracketing cover of $\mathcal{G}@P@\varepsilon$ if for all $g \in \mathcal{G}$, there exists $j \in [m]$ such that:

1. $g_j \leq g$,
2. $Pg \leq Pg_j + \varepsilon$.

Note that $g_1, ..., g_m$ is not necessarily in $\mathcal{G}$.

Theorem 6.6. *Let $\mathcal{G} \subset [0, 1]^{\mathcal{Z}}$, $P \in \mathcal{M}_1(\mathcal{Z})$ and $Z_1, ..., Z_n \sim P$ for $n \in \mathbb{N}$. For all $\varepsilon > 0$, $\delta \in (0, 1)$ and $g \in \mathcal{G}$, it follows that w.p. $1 - \delta$,*

$$Pg - P_n g \leq \inf_{\varepsilon>0} \left[\varepsilon + \sqrt{\frac{\log(N_\varepsilon/\delta)}{2n}} \right],$$

where for all $\varepsilon > 0$,

$$N_\varepsilon = \min\{n \in \mathbb{N} : \textit{ there exists } g_1, ..., g_n \textit{ such that } (g_1, ..., g_n) \textit{ is a lower bracketing cover of } \mathcal{G}@P@\varepsilon\}$$

Proof. Fix an $\varepsilon > 0$. Let $m = N_\varepsilon$ and $g_1, ..., g_m$ be a lower bracketing cover of $\mathcal{G}@P@\varepsilon$. Using additive Chernoff bound, we have that w.p. at least $1-\delta$, it follows that

$$Pg_j \le P_n g_j + \sqrt{\frac{\log(N_\varepsilon/\delta)}{2n}}. \tag{6.2}$$

Pick $g \in \mathcal{G}$ and by definition of lower bracketing cover, there exists $j \in [m]$ such that

$$\begin{aligned} Pg \le Pg_j + \varepsilon \le P_n g_j + \varepsilon + \sqrt{\frac{\log(N_\varepsilon/\delta)}{2n}} && \text{(Definition 6.5(1) and Eq. (6.2))} \\ \le P_n g + \varepsilon + \sqrt{\frac{\log(N_\varepsilon/\delta)}{2n}}. && \text{(Definition 6.5(2))} \end{aligned}$$

Since ε was arbitrary, we then take the infimum over ε:

$$Pg \le P_n g + \inf_{\varepsilon>0}\left[\varepsilon + \sqrt{\frac{\log(N_\varepsilon/\delta)}{2n}}\right].$$

□

Corollary 6.7. *Let $\hat{g}_n = \arg\min_{g\in\mathcal{G}} P_n g$ be the empirical risk minimizer, then it follows that w.p. at least $1-\delta$:*

$$P\hat{g}_n \le \inf_{g\in\mathcal{G}} Pg + 2\inf_{\varepsilon}\left[\varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}}\right]$$

Proof. Fix an $\varepsilon > 0$, by definition of infimum, there exists a g_ε such that

$$Pg_\varepsilon \le \inf_{g\in\mathcal{G}} Pg + \varepsilon \tag{6.3}$$

Denote the lower bracketing cover of $\mathcal{G}@P@\varepsilon =: C_{LB}(G, P, \varepsilon)$. Let $U(\delta, g_\varepsilon, C_{LB}(G, P, \varepsilon))$ be:

$$U(\delta, g_\varepsilon, C_{LB}(G,P,\varepsilon)) := \left\{\forall g \in C_{LB}(G,P,\varepsilon) : Pg \le P_n g + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}}\right\} \cup \left\{P_n g_\varepsilon \le Pg_\varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}}\right\}.$$

Then $U(\delta, g_\varepsilon, C_{LB}(G, P, \varepsilon))$ holds w.p. $1-\delta$. On $U(\delta, g_\varepsilon, C_{LB}(G, P, \varepsilon))$, we have that there exists a $j \in [m]$ such that:

$$\begin{aligned} P\hat{g}_n &\le Pg_j + \varepsilon && \text{(Defn. of lower bracketing cover)} \\ &\le P_n g_j + \varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}} && \text{(Chernoff's bound)} \\ &\le P_n \hat{g}_n + \varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}} && \text{(Defn. of lower bracketing cover)} \\ &\le P_n g_\varepsilon + \varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}} && \text{(Defn. of } \hat{g}_n\text{)} \\ &\le Pg_\varepsilon + \varepsilon + 2\sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}} && \text{(Chernoff's bound)} \\ &\le \inf_{g\in\mathcal{G}} Pg_\varepsilon + 2\varepsilon + 2\sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}} && \text{(Eq. (6.3))} \end{aligned}$$

□

Since ε was arbitrary, we then take the infimum over ε:

$$P\hat{g}_n \leq \inf_{g\in\mathcal{G}} Pg + 2\inf_{\varepsilon}\left[\varepsilon + \sqrt{\frac{\log((N_\varepsilon+1)/\delta)}{2n}}\right]$$

Similarly, using the multiplicative Chernoff bound, we can get the following corollary:

Corollary 6.8. *Let $\hat{g}_n = \arg\min_{g\in\mathcal{G}} P_n g$ be the empirical risk minimizer, then it follows that w.p. at least $1-\delta$:*

$$P\hat{g}_n \leq \inf_{g\in\mathcal{G},\varepsilon>0}\left[Pg + 2\varepsilon + \sqrt{\frac{2Pg\log((N_\varepsilon+1)/\delta)}{2n}} + \sqrt{\frac{P\hat{g}_n\log((N_\varepsilon+1)/\delta)}{2n}} + \frac{\log((N_\varepsilon+1)/\delta)}{3n}\right]$$